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ANALYSIS OF MULTIPLE DELAMINATION IN VISCOELASTIC MULTILAYERED BEAMS WITH RECTANGULAR SECTION SUBJECTED TO PURE TORSION

Victor Rizov*

University of Architecture, Civil Engineering and Geodesy, Sofia, Bulgaria

Abstract. *The properties of continuously inhomogeneous materials depend on the location. This paper is focussed on the problem of multiple delamination in viscoelastic multilayered rectangular beams under loading that induces torsion. Beam layers are made by different materials that are continuously inhomogeneous along the length. General approach for deducing the strain energy release rate (SERR) is developed. The approach considers a beam made of adhesively bonded inhomogeneous layers with parallel delamination cracks. The approach uses the compliance of the beam. The external torques applied on the beam is smooth functions of time. The internal torques are determined by solving an internal statically undeterminate beam structure. Analysis of the SERR in a cantilever beam structure with two parallel delaminations is presented. The SERR found by the general approach is checked by a method that is based on the energy balance in the beam.*

Keywords: *Inhomogeneous Beam, Viscoelastic Behavior, Delamination, Torsion*

1. INTRODUCTION

The properties of continuously inhomogeneous materials depend on the location. Kinds of such materials are functionally graded materials (FGM) which have grabbed the attention of both engineers and scientists around the globe in the recent decades [1] – [3]. In essence, the FGM represent advanced composites which have two or more phases. The microstructure of FGM varies continuously in the solid (this is achieved by gradually varying the ratio of phases during manufacturing) [4], [5]. In this way, the distribution of macroscopic material properties is manipulated to achieve a better performance of the FGM structural member under loadings [6], [7]. The multilayered inhomogeneous material systems consist of different adhesively bonded layers [8] – [14]. These systems are widely used in numerous structural applications in up-to-date engineering due to their superior properties [15] – [19]. Novel applications of various advanced methods and approaches such as the p-version of the finite element method for analyzing the mechanical response of inhomogeneous structures have been reported recently in the research literature [20], [21].

Failure due to delamination is one of the main disadvantages of multilayered systems. Despite extensive research on delamination, there are still problems that have not been sufficiently studied. In particular, delamination under torsional loading requires more attention.

A test method for experimental investigation of delamination fracture in fiber reinforced composites subjected to torsion has been developed in [22]. The specimen has been analyzed using the plate theory. Delamination behaviour has been studied appliying the principles of linear-elastic fracture mechasnics. The fracture toughness has been determined by the compliance calibrtion method.

Mid-plane dellamination behaviour of a symmtric composite laminate subjected to torsion loading has been analysed in [23]. Closed form solutions for the SERR have been derived. The classical lamination theory has been applied for studying the composite laminate.

Delamination behaviour of multilayered coatings under torsion has been invistigated experimentaly in [24]. For this purpose, a new test method has been developed. The test uses an increasing torque that generates loading conditions for delamination fracture.

A glue laminated timber beam subjected to torsional loading is studied experimentaly in [25]. Reinforced concrete blocks at the ends of the timber beam are used for applying the loading during the testing. Important features of the beam behaviour under torsion are exploared.

This paper considers the multiple delamination in viscoelastic multilayered inhomogeneous beams with rectangular sections. The beams are under external torques which changes smoothly with time. The reason for writing this paper is that the previous studies on delamination in multilayered rectangular viscoelastic beams under torsion usually are concerned with beams having one delamination [22] - [27]. In this paper, a general approach for deriving of SERR in beams with any number of parallel delaminations is developed by analyzing the compliance of viscoelastic multilayered beams under pure torsion. The approach is used to investigate SERR in a cantilever with two delamination cracks.

* E-mail of the corresponding author: v_rizov_fhe@uacg.bg

The balance of the strain energy is analyzed for verification.

2. General approach

The beam in Fig. 1 has n longitudinal adhesively bonded layers of viscoelastic materials.

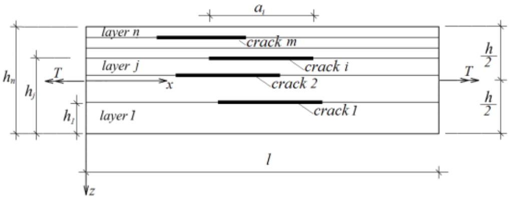

Figure 1. Multilayered viscoelastic beam structure with any number of delamination cracks loaded in torsion

The width and thickness of the beam section are b and h. The longitudinal size of the beam is l. The beam is loaded at its ends by two external torques, T, as shown in Fig. 1. These torques change smoothly over time, t. There are m delamination cracks randomly located between the layers. It should be emphasized that $m < n$ so there is no situation with several delaminations on one interface. The longitudinal size of the *i*-th delamination is a_i. The layers of the beam are of different materials. Furthermore, the layers have different thicknesses.

The behavior of the *j*-th layer is modeled by using shear modulus, G_{jf}, that is time-dependent. Layers of the beam are continuously inhomogeneous in longitudinal direction (the properties of layers depend on abscissa, x (Fig. 1)).

The compliance, C, is analyzed to obtain general solution of SERR for the *i*-th delamination (Fig. 1). C is found as written below

$$C = \frac{\varphi}{T}, \tag{1}$$

where φ is the twist angle of section with torque, T. The angle of twist is found by the theorem of Castiliano

$$\varphi = \frac{\partial U}{\partial T}, \tag{2}$$

where the strain energy, U, is

$$U = \sum_{w=1}^{w=p} \int_{l_w} \frac{T_w^2}{2S_w} dx . \tag{3}$$

In (3), p is the beam portions number, l_w, T_w and S_w are the size, torque and stiffness in torsion of the beam portion. The stiffness in torsion is written as [28]

$$S_w = \frac{8}{\pi^2} \sum_{k=1,3,\ldots}^{\infty} \frac{1}{k^2} \left\{ \frac{\alpha_k}{2} \sum_{j=1}^{n} r_{k,j}\left(h_j - h_{j-1}\right) + \sum_{j=1}^{n} P_{k,j} sh\frac{\alpha_k\left(h_j + h_{j-1}\right)}{2} sh\frac{\alpha_k\left(h_j - h_{j-1}\right)}{2} + \sum_{j=1}^{n} Q_{k,j} sh\frac{\alpha_k\left(h_j - h_{j-1}\right)}{2} ch\frac{\alpha_k\left(h_j + h_{j-1}\right)}{2} \right\}, \tag{4}$$

where

$$\alpha_k = \frac{k\pi}{b}, r_{k,j} = \frac{8G_{jf}b^2}{k^3\pi^3}. \tag{5}$$

The quantities, h_j, are defined in Fig. 1. The following recurrent expressions are used to determine the quantities, $P_{k,j}$ and $Q_{k,j}$ [28]:

$$P_{k,j} = \frac{2}{\left(g_{j,j+1} - 1\right) sh2\alpha_k h_j} \left[Q_{k,j+1} g_{j,j+1} + Q_{K,j} \left(sh^2\alpha_k h_j - g_{j,j+1} ch^2\alpha_k h_j \right) + g_{j,j+1}\left(r_{k,j+1} - r_{k,j}\right) ch\alpha_k h_j) \right], \tag{6}$$

$$P_{k,j+1} = \frac{2}{\left(g_{j,j+1} - 1\right) sh2\alpha_k h_j} \left[Q_{K,j+1} \left(ch^2\alpha_k h_j - g_{j,j+1} sh^2\alpha_k h_j \right) - Q_{k,j} + \left(r_{k,j+1} - r_{k,j}\right) ch\alpha_k h_j) \right], \tag{7}$$

where

$$g_{j,j+1} = \frac{G_{jf}}{G_{(j+1)f}},$$

$$P_{k,1} = -\frac{Q_{k,1} ch\alpha_k h_0 + r_{k,1}}{sh\alpha_k h_0},$$

$$P_{k,n} = -\frac{Q_{k,n} ch\alpha_k h_n + r_{k,n}}{sh\alpha_k h_n}. \tag{8}$$

Here, $j = 1, 2, \ldots, n-1$. By using equations (5) - (8), one can obtain consecutively all unknowns with the same index, k.

Besides the torques in the un-cracked beam portions, the torques in the arms of cracks contribute also to U. It should be mentioned that the torques in the arms of cracks are unknown. In order to determine the torques, the beam under consideration is solved as a statically undetermined system. The torques, T_i, where $i = 1,2,\ldots m$ in the arms of cracks are obtained by solving the indeterminate problem by applying the Menabrea's theorem

$$\frac{\partial U}{\partial T_i} = 0, i = 1,2,\ldots m. \tag{9}$$

From statics we have

$$\sum_{i=1}^{i=m+1} T_i = T \,. \tag{10}$$

Equations (9) and (10) are used to obtain the torques in the arms of cracks. Then, the angle of twist and the compliance can be calculated by applying (1) and (2), respectively.

The SERR, G, for the i-th delamination can be found as

$$G = \frac{T^2}{2b}\frac{dC}{da_i} \,. \tag{11}$$

3. EXAMPLE

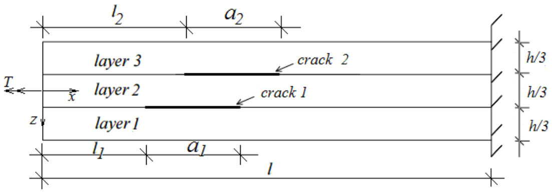

Figure 2. Cantilever beam with two delaminations under torsion

The general approach for deriving SERR for delaminations in multilayered viscoelastic rectangular beams subjected to pure torsion developed in previous section is used to analyze the cantilever beam system in Fig. 2. Formula (12) describes the variation of the torque, T, with time, t

$$T = v_T t \,, \tag{12}$$

where v_T is a parameter that controls the loading. The beam has two cracks from delamination (Fig. 2).

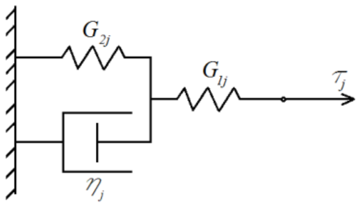

Figure 3. Viscoelastic mechanical model

The viscoelastic behavior of the j-th layer is represented by the mechanical model in Fig. 3. The shear stress, τ_j, applied on the model changes with time as

$$\tau_j = v_j t \,, \tag{13}$$

where v_j is a parameter.

Formula (14) represents the stress-strain-time dependence for the model in Fig. 3 [29]

$$\gamma_j = \frac{v_j t}{G_{j*}} + \frac{q_j v_j}{G_{j*}}\left(1 - \frac{G_{1j}}{G_{j*}}\right)\left(1 - e^{-\frac{G_{*j} t}{q_j G_{1J}}}\right), \tag{14}$$

$$q_j = \frac{\eta_j}{G_{1j} + G_{2j}} \,, \; G_{j*} = \frac{G_{1j} G_{2j}}{G_{1j} + G_{2j}} \,. \tag{15}$$

In formulae (14) and (15), γ_j is the shear strain, G_{1j} and G_{2j} are the moduli of the springs, η_j is the dashpot coefficient of viscosity of the model (Fig. 3). The shear modulus, G_{jf}, of the j-th layer is found as

$$G_{jf} = \frac{\tau_j}{\gamma_j} \,. \tag{16}$$

By using of (13), (14) and (16), one derives

$$G_{jf} = \left[\frac{1}{G_{j*}} + \frac{q_j}{t G_{j*}}\left(1 - \frac{G_{1j}}{G_{j*}}\right)\left(1 - e^{-\frac{G_{*j} t}{q_j G_{1J}}}\right)\right]^{-1} . \tag{17}$$

The modulus (17) is used when determining SERR.

Since layers (Fig. 2) are continuously inhomogeneous in longitudinal direction, the properties, G_{1j}, G_{2j} and η_j, of mechanical model in Fig. 3 change smoothly along the beam.

Functions (18) are used to represent the change in material properties of the j-th layer in the longitudinal direction:

$$G_{1j} = G_{B1j} + \frac{G_{D1j} - G_{B1j}}{l^{\beta_j}} x^{\beta_j} \,,$$

$$G_{2j} = G_{B2j} + \frac{G_{D2j} - G_{B2j}}{l^{\delta_j}} x^{\delta_j} \,,$$

$$\eta_j = \eta_{Bj} + \frac{\eta_{Dj} - \eta_{Bj}}{l^{\lambda_j}} x^{\lambda_j} \,. \tag{18}$$

where G_{B1j}, G_{B2j} and η_{Bj} are the values of G_{1j}, G_{2j} and η_j at the left-hand end of the beam.

The values of G_{1j}, G_{2j} and η_j at the right-hand end of the beam are denoted by G_{D1j}, G_{D2j} and η_{Dj}, respectively. The parameters, β_j, δ_j and λ_j, control the distribution of G_{1j}, G_{2j} and η_j along the length of the beam.

The SERR for the two delaminations in Fig. 2 is determined by applying (11). The SERR is a function of time due the viscoelastic behavior of the beam. Another reason for dependency of the SERR on time is the fact that the external torque varies with time.

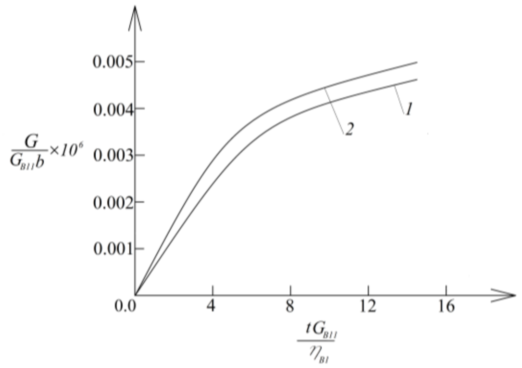

Figure 4. The SERR - time curves (1 – for delamination 1 and 2 – for delamination 2)

To check (11), the SERR is deduced also by analyzing the balance of the strain energy. The SERR is written as

$$G = \frac{1}{b}\left(T\frac{\partial \varphi}{\partial a_i} - \frac{\partial U}{\partial a_i} \right). \quad (19)$$

The SERR derived by (19) is identical to that obtained by using (11) which is a control of the analysis described in this work.

Another check is performed by applying an approach that obtains the SERR as the difference between the strain energies in the crack arms behind the crack front and the strain energy in the beam ahead of the crack front [30]. This approach leads to Eq. (20) for the SERR in the beam under consideration.

$$G = \frac{T_{wl}^2}{2bS_{wl}} + \frac{T_{wu}^2}{2bS_{wu}} - \frac{T_w^2}{2bS_w}, \quad (20)$$

where T_{wl} and T_{wu} are the torsion moments in the lower and upper crack arms, S_{wl} and S_{wu} are the stiffnesses in torsion of the lower and upper crack arms. Equation (20) yields results that are identical to these found by (11).

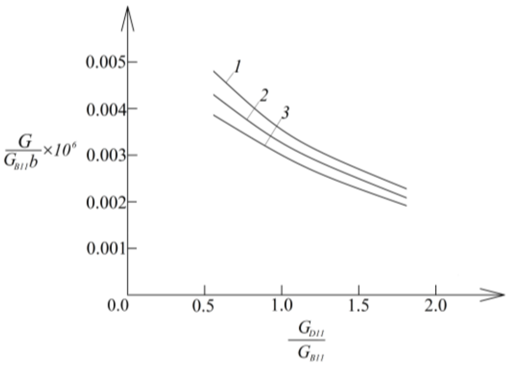

Figure 5. The SERR - G_{D11}/G_{B11} ratio curves (1 – at $\eta_{D1}/\eta_{B1} = 0.5$, 2 – at $\eta_{D1}/\eta_{B1} = 1.0$ and 3 – at $\eta_{D1}/\eta_{B1} = 2.0$)

A parametric study of SERR for two delaminations in the beam configuration with rectangular section under pure torsion shown in Fig. 2 is carried-out.

The parametric study results are obtained for $b = 0.030$ m, $h = 0.040$ m, $l_1 = 0.200$ m, $l_2 = 0.300$ m, $l = 0.800$ m, $a_1 = 0.250$ m, $a_2 = 0.350$ m, $\beta_j = 0.6$, $\delta_j = 0.7$, $\lambda_j = 0.8$, $j = 1,2,3$ and $v_T = 0.4 \times 10^{-6}$ Nm/s.

The SEER – time curves for both delamination cracks are plotted in Fig. 4 . The SERR and time are expressed in dimensionless form as $G_N = G/(G_{B11}b)$ and $t_N = tG_{B11}/\eta_{B1}$. The curves in Fig. 4 indicate growth of the SERR with time for both delaminations. The explanation for this growth lies in the viscoelastic behavior of the beam structure and the increase in external torque.

The change of SERR due to inhomogeneity of layer 1 of the beam is investigated. The SERR - G_{D11}/G_{B11} ratio curves at three η_{D1}/η_{B1} ratios are plotted in Fig. 5. It is evident from Fig. 5 that when G_{D11}/G_{B11} ratio grows, the SERR reduces. An increase in the ratio, η_{D1}/η_{B1}, leads to a decrease in SERR (Fig. 5). The reason for this behavior is the increase of the beam stiffness.

4. Conclusion

A general approach for deducing SERR in viscoelastic multilayered beams with any number of delamination cracks loaded in pure torsion is developed. The approach is applied to derive SERR in a beam having two parallel delaminations. The study shows that:

1) SERR grows with time for both delaminations;
2) SERR reduces when G_{D11}/G_{B11} ratio increases;
3) Increase of η_{D1}/η_{B1} ratio results in decrease of SERR.

The main practical applications of this work are in proving the safety of multilayered beam structures with viscoelastic behavior in the case of multiple delamination under torsion. For example, the solution obtained here can be used to check whether delamination cracks in a multilayered beam will begin to propagate by comparing the SERR with the fracture toughness. The advantage of the method developed in this work is that it represents a unified approach for obtaining SERR in multilayered beams with an arbitrary number of delaminations.

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