RAD Conference Proceedings, vol. 8, pp. 16-19, 2024
ISSN 2466-4626 (online) | DOI: 10.21175/RadProc.2024.04
www.rad-proceedings.org

FUNCTIONALLY GRADED FRAMES UNDER SUPPORT DISPLACEMENTS: A LONGITUDINAL FRACTURE ANALYSIS WITH REFRENCE TO NON-LINEAR RELAXATION

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Abstract. *The current study deals with the problem of longitudinal fracture in functionally graded load-caring frame structures under support displacements in the conditions of non-linear relaxation behavior. The latter is taken in account by applying a non-linear stress-strain-time constitutive law that holds for viscoelastic engineering materials subjected to constant strains. The frame under consideration is functionally graded along its thickness (thus, the material properties vary continuously along the thickness of the frame members). The frame is statically undetermined. Therefore, the support displacements induce stresses in the frame. These stresses lead to longitudinal fracture in the frame that is analyzed theoretically. The time-dependent strain energy release rate (SERR) for a longitudinal crack in the frame is derived by considering the energy balance under non-linear relaxation. The time-dependent complementary strain energy in the frame is analyzed for verifying the solution of the SERR due to support displacements. Various graphs are plotted to illustrate the effects of different factors (magnitude of support displacements, time, etc.) on the longitudinal fracture behavior. Analyzing the combined effects of static indeterminacy, support displacements and non-linear relaxation behavior on longitudinal fracture of functionally graded frame structures is the main novelty and the added value of the current paper.*

Keywords: *Frame, Support Displacement, Longitudinal Fracture, Relaxation, Structure*

1. INTRODUCTION

The part of structures and mechanisms in different sectors of modern engineering made by functionally graded materials increases constantly. These materials have properties which vary continuously along certain directions in the body of the structure [1]-[9]. Applications of functionally graded materials in sandwich plates are considered in [7]. The influence of porosity on stability behavior is analyzed [7]. The use of functionally graded materials in sandwich plates lying on elastic foundations is studied in [8]. Free vibration behavior of functionally graded plates on a viscoelastic foundation is investigated in [9]. The property variation can be designed to amend the performance and efficiency of structures [10]-[13]. The increased demand on functionally graded materials leads to quick development of various technologies for building-up functionally graded structures [14]-[15].

One of the frequent causes of structural damage in functionally graded engineering materials built-up layer by layer is the longitudinal fracture, i.e. formation of cracks longitudinally to the layers [16]. These cracks may cause severe degradation of the structure and even structural collapse with lose of human lives. The occurrence of longitudinal fracture depends on many factors (mechanical behavior of the functionally graded material, structural geometry, effects of the environment, external influences and loads on the structure, etc.). In principle, a prediction of the phenomena of longitudinal fracture requires calculation of some parameters. Among them, the SERR is one of the most useable [16]-[19]. Therefore, analyzing of the SERR has a fundamental sense for design of safe structures [17]-[19].

In this paper, the problem of longitudinal fracture in functionally graded load-caring frame structures under support displacements in the conditions of non-linear relaxation behavior is treated. The structures are statically undetermined. Namely this leads to appearance of stresses when the structures are under support displacements. The practice indicates that the statically indeterminate frames in various load-carrying engineering applications are frequently under support displacements. Also, often these frames have non-linear relaxation behavior. Analyzing the combined effects of the static indeterminacy and the non-linear relaxation behavior represents a scientific interest and is the objective of this paper. The SERR for the longitudinal crack is derived accounting for the non-linear relaxation behavior of the frame. A method based on consideration of the energy balance is applied for obtaining the SERR. The solution is verified by using a method that derives the SERR by differentiating the complementary strain energy with respect to the area of the crack. It is analyzed how the SERR is affected by some relevant factors for the problem under consideration, namely support displacement value, time, and relative crack length.

2. THEORETICAL ANALYSIS

Figure 1 shows schema of the load-carrying frame structure examined in this paper. The frame hosts a longitudinal crack in portion, D_3D_4. The material of the frame is functionally graded along the thickness. The frame is built-up in section, D_1. Section, D_5, is supported by a roller (Figure 1). This means that the frame represents a structure having one degree of static

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indeterminacy. Section, D_5, undergoes a vertical support displacement, w_{D5}, as shown in Figure 1. This generates stresses in the frame that influence the longitudinal crack. This influence is examined here with considering the effect of non-linear relaxation behavior. Equation (1) presents non-linear stress-stain-time relationship that is applied for treating the relaxation [16].

$$\sigma = \frac{B\varepsilon}{\left[tHL^{n-1}(n-1)\varepsilon^{n-1}+1\right]^{\frac{1}{n-1}}} \tag{1}$$

where t is time, B, H, L and n are material parameters, σ is stress, ε is strain.

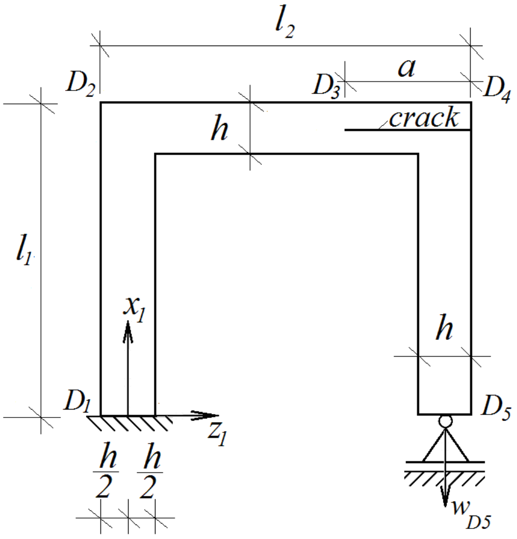

Figure 1. Frame under support displacement.

Equations (2), (3), (4) and (5) describe the change of B, H, L and n along the thickness.

$$B = B_{ou} + \frac{B_{in} - B_{ou}}{h^{\alpha}}\left(\frac{h}{2} + z_1\right)^{\alpha} \tag{2}$$

$$H = H_{ou} + \frac{H_{in} - H_{ou}}{h^{\beta}}\left(\frac{h}{2} + z_1\right)^{\beta} \tag{3}$$

$$L = L_{ou} + \frac{L_{in} - L_{ou}}{h^{\delta}}\left(\frac{h}{2} + z_1\right)^{\delta} \tag{4}$$

$$n = n_{ou} + \frac{n_{in} - n_{ou}}{h^{\eta}}\left(\frac{h}{2} + z_1\right)^{\eta} \tag{5}$$

where

$$-\frac{h}{2} \le z_1 \le \frac{h}{2}. \tag{6}$$

Here, the values of the material parameters, B, H, L and n, at the outer surface of the frame are B_{ou}, H_{ou}, L_{ou} and n_{ou}. The values of the material parameters at the inner surface of the frame are B_{in}, H_{in}, L_{in} and n_{in}. The quantities, α, β, δ and η, control the change of B, H, L and n, respectively.

The mechanical response of the frame to the support displacement, w_{D5}, is treated by the following equations. First, Equation (7) is composed by expressing the support displacement as function of curvatures, κ_{D1D2}, κ_{D2D3}, κ_{D3D4} and κ_{D4D5}, of the frame portions.

$$\begin{aligned} w_{D5} = & \int_0^{l_1-\frac{h}{2}} \kappa_{D1D2}(l_2 - h)dx_1 + \\ & \int_0^{l_2-\frac{h}{2}-a} \kappa_{D2D3}(l_2 - h - x_2)dx_2 + \\ & \int_0^{a-\frac{h}{2}} \kappa_{D3D4}\left(a - \frac{h}{2} - x_3\right)dx_3 + \\ & \int_0^{l_1-\frac{h}{2}} (-\kappa_{D4D5}z_{4n})dx_4 \end{aligned} \tag{7}$$

where a is the crack length, z_{4n} is the location of the neutral axis in frame portion, D_4D_5. Equation (7) is derived by using the integrals of Maxwell-Mohr [20].

Equation (8) is obtained by relating the axial force, N_{D1D2}, in frame portion, D_1D_2, with the stress, σ_{D1D2}.

$$N_{D1D2} = \iint_{(A)} \sigma_{D1D2} dA \tag{8}$$

where σ_{D1D2} is related with strain, ε_{D1D2}, via Eq. (1). The strain distribution is presented by Eq. (9).

$$\varepsilon_{D1D2} = \kappa_{D1D2}(z_1 - z_{1n}) \tag{9}$$

Equation (10) relates the bending moment, M_{D1D2}, with the stress, σ_{D1D2}.

$$M_{D1D2} = \iint_{(A)} \sigma_{D1D2} z_1 dA \tag{10}$$

The next six equations are obtained by relating the axial forces and the bending moments in frame portions, D_2D_3, D_3D_4 and D_4D_5,, with the corresponding stresses.

$$N_{D2D3} = \iint_{(A)} \sigma_{D2D3} dA, \tag{11}$$

$$M_{D2D3} = \iint_{(A)} \sigma_{D2D3} z_2 dA, \tag{12}$$

$$N_{D3D4} = \iint_{(A)} \sigma_{D3D4} dA, \tag{13}$$

$$M_{D3D4} = \iint_{(A)} \sigma_{D3D4} z_3 dA, \tag{14}$$

$$N_{D4D5} = \iint_{(A)} \sigma_{D4D5} dA, \tag{15}$$

$$M_{D4D5} = \iint_{(A)} \sigma_{D4D5} z_4 dA. \tag{16}$$

Equations (17) – (24) relate the axial forces and the bending moments in the frame portions with the reaction, R_{D5}, in the roller support in section, D_5.

$$N_{D1D2} = -R_{D5} \tag{17}$$

$$M_{D1D2} = R_{D5}(l_2 - h) \tag{18}$$

$$N_{D2D3} = 0 \tag{19}$$

$$M_{D2D3} = R_{D5}(l_2 - x_2 - h) \tag{20}$$

$$N_{D3D4} = 0 \tag{21}$$

$$M_{D3D4} = R_{D5}\left(a - x_3 - \frac{h}{2}\right) \tag{22}$$

$$N_{D4D5} = R_{D5} \tag{23}$$

$$M_{D4D5} = 0 \tag{24}$$

Equations (7) – (24) are used to determine the curvatures, the locations of the neutral axis and the reaction in the roller support. These quantities are needed to derive the SERR, G, for the longitudinal crack in the frame in Fig. 1 via Equation (25).

$$G = \frac{1}{b}\left(R_{D5}\frac{\partial w_{D5}}{\partial a} - \frac{\partial U^*}{\partial a} \right) \tag{25}$$

where the complementary strain energy, U^*, in the frame is found by integrating the specific complementary strain energies in the frame portions. The SERR yielded by Equation (25) is checked-up by differentiating U^* with respect to the crack area.

3. RESULTS

The numerical results are used to plot various graphs in Fig. 2, Fig. 3 and Fig. 4 illustrating how the SERR is affected by different factors (magnitude of support displacements, time, etc.).

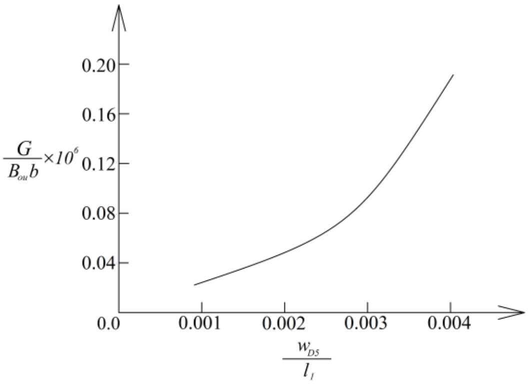

Figure 2. The SERR vs. w_{D5}/l_1 ratio.

These results are derived for $l_1 = 4.000$ m, $l_2 = 3.000$ m, $b = 0.250$ m, $h = 0.300$ m, $a = 1.500$ m, $w_{D5} = 0.016$ m, $\alpha = 0.5$, $\beta = 0.5$, $\delta = 0.6$ and $\eta = 0.6$.

Figure 2 demonstrates the typical process of SERR development when the relative support displacement magnitude, w_{D5}/l_1, changes in the range between 0.0001 and 0.004.

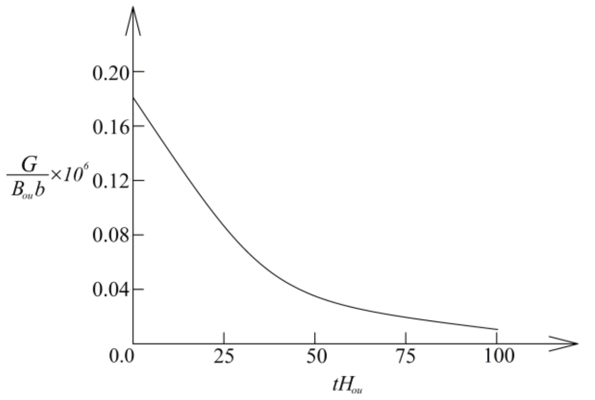

Figure 3. The SERR vs. non-dimensional time.

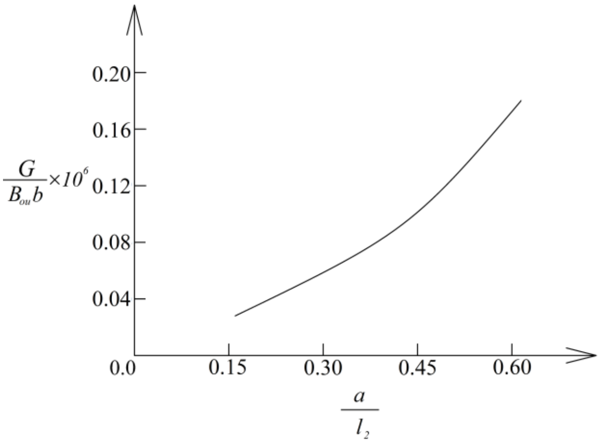

Figure 4. The SERR vs. a/l_2 ratio.

It is to be seen in Fig. 3 that during the increase of w_{D5}/l_1 the SERR continuously grows.

Figure 3 shows the results of investigation of the SERR change with time. This change is induced by the relaxation behavior of the viscoelastic frame under support displacement. It is to be seen that the SERR is essentially reduced in course of time (Fig. 3).

The results of our investigations of the change of the SERR in the frame generated by change of the relative crack length, a/l_2, are shown in Fig. 4. The results demonstrate significant change in the SERR (Fig. 4). The higher the relative crack length the higher is the SERR.

4. CONCLUSION

A theoretical treatment of the question for the longitudinal fracture in functionally graded frames under support displacements when non-linear relaxation takes place is given in the current work. The SERR in the frame induced by the support displacements is determined. On the basis of this solution it can be concluded that the SERR is affected significantly by the two factors (support displacement and non-linear relaxation) under consideration. The functional relation between the SERR and the support displacement, time and relative crack length is non-linear. Growing support displacement causes rise of the SERR. Rapid rise of the SERR is caused also by growth of the relative crack length. The non-linear relaxation causes a continuous drop of the SERR in course of time. In general, these results underline how important is to give full consideration of support displacement and non-linear relaxation when investigating longitudinal fracture in functionally graded frames. Further work is warranted to understand mechanisms of longitudinal fracture under support displacements and non-linear relaxation in frames with more degrees of indeterminacy.

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